Fuzzyfied Differential Geometry

This chapter is concerned with topics on *Differential Geometry*, especially the differential geometry of closed curves or contours. Contours may be obtained routinely from the processing of black and white images, as piecewise continuous curves. These curves may be *smoothed* by essentially the same techniques as have been applied before, namely convolution with Gauss functions. This leads to nice formulas for the difference between the discrete and the continuous in the neighbourhood of a *kink* and an elegant formula for the overall discrepancy between the original and the fuzzyfied curve in smooth regions of the former.

Subsequent Fuzzyfications

The *fuzzyfication* of a two-dimensional curve is defined as follows:

$$\begin{bmatrix} \overline{x}(t) \\ \overline{y}(t) \end{bmatrix} = \int_{-\infty}^{+\infty} \begin{bmatrix} x(\tau) \\ y(\tau) \end{bmatrix} \frac{1}{\sigma\sqrt{2\pi}} \, e^{-\frac{1}{2}(t-\tau)^2/\sigma^2} \, d\tau$$

This is the general definition. Now σ is replaced by σ_1, because any subsequent fuzzyfication of a curve, if it has been fuzzyfied already, should be defined by:

$$\begin{bmatrix} \overline{\overline{x}}(t) \\ \overline{\overline{y}}(t) \end{bmatrix} = \int_{-\infty}^{+\infty} \begin{bmatrix} \overline{x}(u) \\ \overline{y}(u) \end{bmatrix} \frac{1}{\sigma_2\sqrt{2\pi}} \, e^{-\frac{1}{2}(t-u)^2/\sigma_2^2} \, du$$

$$= \int_{-\infty}^{+\infty} \left\{ \int_{-\infty}^{+\infty} \begin{bmatrix} x(\tau) \\ y(\tau) \end{bmatrix} \frac{1}{\sigma_1\sqrt{2\pi}} \, e^{-\frac{1}{2}(u-\tau)^2/\sigma_1^2} \, d\tau \right\} \frac{1}{\sigma_2\sqrt{2\pi}} \, e^{-\frac{1}{2}(t-u)^2/\sigma_2^2} \, du$$

$$= \frac{1}{\sigma_1\sqrt{2\pi}} \, \frac{1}{\sigma_2\sqrt{2\pi}} \int_{-\infty}^{+\infty} \begin{bmatrix} x(\tau) \\ y(\tau) \end{bmatrix} \left\{ \int_{-\infty}^{+\infty} e^{-\frac{1}{2}(u-\tau)^2/\sigma_1^2} \, e^{-\frac{1}{2}(t-u)^2/\sigma_2^2} \, du \right\} d\tau$$

The integral between { } is worked out separately:

$$= \int_{-\infty}^{+\infty} e^{-\frac{1}{2}\left[(u-\tau)^2/\sigma_1^2 + (t-u)^2/\sigma_2^2\right]} \, du$$

Restricting attention to the exponent:

$$(u-\tau)^2/\sigma_1^2 + (u-t)^2/\sigma_2^2 = w_1(u-\tau)^2 + w_2(u-t)^2$$

Where $w_1 = 1/\sigma_1^2$ and $w_2 = 1/\sigma_2^2$. Continuing:

$$w_1(u-\tau)^2 + w_2(u-t)^2 = w_1u^2 - 2w_1\tau u + w_1\tau^2 + w_2u^2 - 2w_2tu + w_2t^2 =$$

$$(w_1+w_2)\left[u^2 - 2\frac{w_1\tau + w_2t}{w_1+w_2}u + \left(\frac{w_1\tau + w_2t}{w_1+w_2}\right)^2\right]$$

$$-\frac{(w_1\tau + w_2 t)^2}{w_1+w_2} + \frac{(w_1\tau^2 + w_2 t^2)(w_1+w_2)}{w_1+w_2} =$$

$$(w_1+w_2)\left[u - \frac{w_1\tau + w_2 t}{w_1+w_2}\right]^2$$

$$-\frac{w_1^2\tau^2 + 2w_1w_2\tau t + w_2^2t^2}{w_1+w_2} + \frac{w_1^2\tau^2 + w_1w_2\tau^2 + w_2^2t^2 + w_1w_2t^2}{w_1+w_2} =$$

$$(w_1+w_2)\left[u - \frac{w_1\tau + w_2 t}{w_1+w_2}\right]^2 + \frac{w_1w_2}{w_1+w_2}(\tau^2 - 2\tau t + t^2)$$

The result is kind of a *Lemma*:

$$w_1(u-\tau)^2 + w_2(u-t)^2 = (w_1+w_2)\left[u - \frac{w_1\tau + w_2 t}{w_1+w_2}\right]^2 + \frac{w_1w_2}{w_1+w_2}(\tau - t)^2$$

Now re-substitute $w_1 = 1/\sigma_1^2$ and $w_2 = 1/\sigma_2^2$. Then:

$$w_1 + w_2 = \frac{\sigma_1^2 + \sigma_2^2}{\sigma_1^2\sigma_2^2}$$

$$\frac{w_1\tau + w_2 t}{w_1+w_2} = \left[\frac{1}{\sigma_1^2}\tau + \frac{1}{\sigma_2^2}t\right]\frac{\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2} = \frac{\sigma_2^2\tau + \sigma_1^2 t}{\sigma_1^2+\sigma_2^2}$$

$$\frac{w_1w_2}{w_1+w_2} = \frac{1}{\sigma_1^2\sigma_2^2}\left[\frac{\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2}\right] = \frac{1}{\sigma_1^2+\sigma_2^2}$$

Hence:

$$w_1(u-\tau)^2 + w_2(u-t)^2 = \frac{\sigma_1^2+\sigma_2^2}{\sigma_1^2\sigma_2^2}\left[u - \frac{\sigma_2^2\tau + \sigma_1^2 t}{\sigma_1^2+\sigma_2^2}\right]^2 + \frac{(\tau - t)^2}{\sigma_1^2+\sigma_2^2}$$

Giving for the integral between { } :

$$\int_{-\infty}^{+\infty} e^{-\frac{1}{2}\left[(u-\tau)^2/\sigma_1^2 + (t-u)^2/\sigma_2^2\right]}\, du =$$

$$\int_{-\infty}^{+\infty} exp\left(-\frac{1}{2}\left[u - \frac{\sigma_2^2\tau + \sigma_1^2 t}{\sigma_1^2+\sigma_2^2}\right]^2 \Big/ \frac{\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2}\right)\, du \;.\; exp\left(-\frac{1}{2}\frac{[\tau - t]^2}{\sigma_1^2+\sigma_2^2}\right) =$$

$$\sqrt{2\pi\frac{\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2}}\; e^{-\frac{1}{2}(\tau-t)^2/(\sigma_1^2+\sigma_2^2)}$$

Substitute this into the expression with the double integral:

$$\frac{1}{\sigma_1\sqrt{2\pi}}\,\frac{1}{\sigma_2\sqrt{2\pi}}\int_{-\infty}^{+\infty}\begin{bmatrix} x(\tau) \\ y(\tau)\end{bmatrix}\left\{\sqrt{2\pi\frac{\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2}}\; e^{-\frac{1}{2}(\tau-t)^2/(\sigma_1^2+\sigma_2^2)}\right\}\, d\tau$$

$$= \int_{-\infty}^{+\infty} \begin{bmatrix} x(\tau) \\ y(\tau) \end{bmatrix} \frac{1}{\sqrt{2\pi(\sigma_1^2+\sigma_2^2)}} \; e^{-\frac{1}{2}(t-\tau)^2/(\sigma_1^2+\sigma_2^2)} \; d\tau$$

Conclusion:

$$\begin{bmatrix} \overline{\overline{x}}(t) \\ \overline{\overline{y}}(t) \end{bmatrix} = \int_{-\infty}^{+\infty} \begin{bmatrix} x(\tau) \\ y(\tau) \end{bmatrix} \frac{1}{\sigma\sqrt{2\pi}} \; e^{-\frac{1}{2}(t-\tau)^2/\sigma^2} \; d\tau \qquad \text{where:} \qquad \sigma = \sqrt{\sigma_1^2+\sigma_2^2}$$

With other words: if a fuzzyfied curve is fuzzyfied again, then the result is a fuzzyfied curve, where the spreads are added according to Euclid's theorem.

Ideal and Material Curves

It is asked whether the LineWidth of a non-ideal (thickened) curve in the plane has great influence on its geometrical properties, like the midpoint (center of gravity) or the moments of inertia. Now every real curve can be represented for our purpose as an ideal line, also called *skeleton line*, with a certain thickness or definite linewidth superimposed on it. We can walk back and forth along the skeleton, or or we can take a walk around the contour of the thickened curve. What then will be the difference between the idealized (thinned) curve and the *materialized* (thickened) one ?
The difference in their momenta can be readily calculated. Let the linewidth be given by D and $(cos(\phi), sin(\phi))$ the normal vector at the skeleton $(x(t), y(t))$. Then the first moment of inertia is:

$$\frac{1}{L}\int_0^L \left[\begin{array}{c} x(t) + D/2.cos(\phi(t)) \\ y(t) + D/2.sin(\phi(t)) \end{array} \right] dt =$$

$$\frac{1}{L}\int_0^L \left[\begin{array}{c} x(t) \\ y(t) \end{array} \right] dt + D/2 \frac{1}{L}\int_0^L \left[\begin{array}{c} cos(\phi(t)) \\ sin(\phi(t)) \end{array} \right] dt =$$

$$\frac{1}{L}\int_0^L \left[\begin{array}{c} x(t) \\ y(t) \end{array} \right] dt$$

Because integrating the normal back and forth along the skeleton cancels out the accompanying integrals: every normal has a neighbour in the opposite direction, except at infinitesimal regions around the tips. This means that the midpoint of the materialized curve is merely *identical* to the midpoint of the idealized curve. Now go for some second order moments. A nice one is the artithmetic mean of the two second order moments:

$$\frac{1}{2}\frac{1}{L}\int_0^L \left[x(t) + D/2.cos(\phi(t))\right]^2 + \left[y(t) + D/2.sin(\phi(t))\right]^2 \; dt =$$

$$\frac{1}{2L}\int_0^L \left[x^2(t) + y^2(t)\right] dt + D\frac{1}{2L}\int_0^L \left[x(t).cos(\phi) + y(t).sin(\phi)\right] dt$$

$$+(D/2)^2 \frac{1}{2L} \int_0^L \left[cos^2(\phi) + sin^2(\phi)\right] dt$$

Here the first integral is the mean second order moment of the skeleton line. The second integral is an integral over the inner product of vectors pointing toward the skeleton $(x(t), y(t))$, and the normals $(cos(\phi), sin(\phi))$. In this summation each skeleton point has an inner product with a normal and also with the exact opposite of this normal. These contributions will mutually cancel out. Thus the second integral is zero. The third integral is equal to L. So the mean moment of inertia of a thicked line is equal to the moment of inertia of the thinned line, plus a contribution which equals $(D^2/8)$.
The gist of this result is that influence of the linewidth on geometrical properties of the curve, when compared with the impact $L >> D$ of the size of the thinned line, may often be considered as being of secondary importance.

Signed Curvature

This (sub)section is only for the sake of precision. A quite common mistake in mathematical readings about differential geometry is to define the curvature of a curve as being zero or positive from the start. This is an approach which is to be characterized as highly unsuitable, as will be pointed out in the sequel. Let's repeat a few elementary facts. The arc-length of a curve:

$$s = \int_0^t \sqrt{(x')^2 + (y')^2}\, dt \quad \Longrightarrow$$

$$\frac{ds}{dt} = \sqrt{(x')^2 + (y')^2} \quad \Longrightarrow \quad \frac{dt}{ds} = \frac{1}{\sqrt{(x')^2 + (y')^2}}$$

The tangent at a curve, when differentiated to the arc length:

$$\vec{\iota} = \frac{d}{ds} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \frac{dt}{ds} \frac{d}{dt} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} / \sqrt{(x')^2 + (y')^2}$$

Which implies that the length of the tangent is always unity (equal to 1).
The derivative of the tangent:

$$\frac{d\vec{\iota}}{ds} = \frac{d\vec{\iota}}{dt} \frac{dt}{ds} = \frac{d}{dt} \left\{ \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} / \sqrt{(x')^2 + (y')^2} \right\} \frac{1}{\sqrt{(x')^2 + (y')^2}}$$

Just do it, in two steps:

$$\frac{d}{dt} \left(\frac{x'}{\sqrt{(x')^2 + (y')^2}} \right) =$$

$$\frac{x'' \sqrt{(x')^2 + (y')^2} - x'(2x'x'' + 2y'y'').\frac{1}{2} \left[(x')^2 + (y')^2\right]^{-\frac{1}{2}}}{(x')^2 + (y')^2} =$$

$$\frac{x''\left[(x')^2+(y')^2\right]-x'(x'x''+y'y'')}{\left[(x')^2+(y')^2\right]^{3/2}}=-y'\frac{x'y''-x''y'}{\left[(x')^2+(y')^2\right]^{3/2}}$$

A shortcut for the second step is to exchange x and y in the above:

$$\frac{d}{dt}\left(\frac{y'}{\sqrt{(x')^2+(y')^2}}\right)=-x'\frac{y'x''-y''x'}{\left[(y')^2+(x')^2\right]^{3/2}}=x'\frac{x'y''-x''y'}{\left[(x')^2+(y')^2\right]^{3/2}}$$

Summarizing:

$$\frac{d}{ds}\begin{bmatrix}x'\\y'\end{bmatrix}=\frac{x'y''-x''y'}{\left[(x')^2+(y')^2\right]^{3/2}}\begin{bmatrix}-y'\\x'\end{bmatrix}\frac{1}{\sqrt{(x')^2+(y')^2}}$$

Here the vector with components $(-y',x')/\sqrt{(x')^2+(y')^2}$ is recognized as the unit normal $\vec{n}$. Hence the end-result is:

$$\frac{d\vec{t}}{ds}=\rho\vec{n} \qquad \text{where:} \quad \rho=\frac{x'y''-x''y'}{\left[(x')^2+(y')^2\right]^{3/2}}$$

It is seen that ρ can be positive as well as negative, which is the crux of the matter here.

Fuzzy Kinks

Imagine two arbitrary straight half-lines, which intersect each other at the origin:

$$\begin{bmatrix}x_1(t)\\y_1(t)\end{bmatrix}=-t\begin{bmatrix}cos(\alpha)\\sin(\alpha)\end{bmatrix} \qquad \text{for} \quad -\infty<t\le 0$$

$$\begin{bmatrix}x_2(t)\\y_2(t)\end{bmatrix}=+t\begin{bmatrix}cos(\beta)\\sin(\beta)\end{bmatrix} \qquad \text{for} \quad 0\le t<+\infty$$

The union of these two half-lines is a line-with-a-kink, which is to be studied further:

$$\begin{bmatrix}x(t)\\y(t)\end{bmatrix}=\begin{bmatrix}x_1(t)\\y_1(t)\end{bmatrix}\cup\begin{bmatrix}x_2(t)\\y_2(t)\end{bmatrix}$$

We are curious about the fuzzyfication of the kink (at the origin), which is defined as follows:

$$\int_{-\infty}^{+\infty}\frac{1}{\sigma\sqrt{2\pi}}\,e^{-\frac{1}{2}(t-\tau)^2/\sigma^2}\begin{bmatrix}x(\tau)\\y(\tau)\end{bmatrix}d\tau \qquad \text{for} \quad t=0$$

Due to the above, this integral is equal to the sum of two parts:

$$\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{0}e^{-\frac{1}{2}(t-\tau)^2/\sigma^2}\,(-\tau)\,d\tau\cdot\begin{bmatrix}cos(\alpha)\\sin(\alpha)\end{bmatrix}$$

$$+ \frac{1}{\sigma\sqrt{2\pi}} \int_{0}^{+\infty} e^{-\frac{1}{2}(t-\tau)^2/\sigma^2} \; (+\tau) \; d\tau \cdot \left[\begin{array}{c} cos(\beta) \\ sin(\beta) \end{array} \right]$$

Working out:

$$\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{0} e^{-\frac{1}{2}(t-\tau)^2/\sigma^2} \; (-\tau) \; d\tau =$$

$$\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{0} e^{-\frac{1}{2}(t-\tau)^2/\sigma^2} \; \frac{t-\tau}{\sigma^2} \; d\tau \; . \; \sigma^2 \quad - \quad \frac{t}{\sigma\sqrt{2\pi}} \int_{-\infty}^{0} e^{-\frac{1}{2}(t-\tau)^2/\sigma^2} d\tau$$

Substitute $u = -\frac{1}{2}(t-\tau)^2/\sigma^2$ into the first integral and $v = (\tau - t)/\sigma$ into the second one, then $du = (t-\tau)/\sigma^2 d\tau$ and $u(0) = -\frac{1}{2}t^2/\sigma^2$, $v(0) = -t/\sigma$, giving:

$$= \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{-\frac{1}{2}t^2/\sigma^2} e^u \; du \; . \; \sigma^2 \quad - \quad t \cdot \int_{-\infty}^{-t/\sigma} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}v^2} dv =$$

$$\frac{\sigma}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2/\sigma^2} \; - \; t \; Erf(-t/\sigma)$$

Quite analogously:

$$\frac{1}{\sigma\sqrt{2\pi}} \int_{0}^{+\infty} e^{-\frac{1}{2}(t-\tau)^2/\sigma^2} \; (+\tau) \; d\tau$$

$$= -\frac{\sigma}{\sqrt{2\pi}} \int_{-\frac{1}{2}t^2/\sigma^2}^{-\infty} e^u \; du \; . \; \sigma^2 \quad + \quad t \cdot \int_{-t/\sigma}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}v^2} dv =$$

$$\frac{\sigma}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2/\sigma^2} \; + \; t \; Erf(t/\sigma)$$

Thus the general outcome is:

$$\vec{r}(t) = \left\{ \frac{\sigma}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2/\sigma^2} \; - \; t \; Erf(-t/\sigma) \right\} \cdot \left[\begin{array}{c} cos(\alpha) \\ sin(\alpha) \end{array} \right]$$

$$+ \left\{ \frac{\sigma}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2/\sigma^2} \; + \; t \; Erf(t/\sigma) \right\} \cdot \left[\begin{array}{c} cos(\beta) \\ sin(\beta) \end{array} \right]$$

It is seen that these expressions converge to the original straight half-lines for $|t| \gg \sigma$. When specializing for $t = 0$, we find:

$$= \frac{\sigma}{\sqrt{2\pi}} \left[\begin{array}{c} cos(\alpha) + cos(\beta) \\ sin(\alpha) + sin(\beta) \end{array} \right]$$

This expression can be simplified further by using a few goniometric formulas, or even better by sketching an appropriate figure and using some elementary geometry.

$$cos(\alpha) + cos(\beta) = 2.cos\left(\frac{\alpha+\beta}{2}\right).cos\left(\frac{\alpha-\beta}{2}\right)$$

$$sin(\alpha) + sin(\beta) = 2.sin\left(\frac{\alpha+\beta}{2}\right).cos\left(\frac{\alpha-\beta}{2}\right)$$

Giving for the fuzzyfied kink:

$$\frac{\sigma}{\sqrt{2\pi}}\,.\,2\,.\,cos\left(\frac{\alpha-\beta}{2}\right)\left[\begin{array}{c} cos\left(\frac{\alpha+\beta}{2}\right) \\ sin\left(\frac{\alpha+\beta}{2}\right) \end{array}\right]$$

This is a vector dividing the angle between the lines in two equal halves. And its length is given by the astonishing simple formula:

$$2\,.\,cos(\phi/2)\frac{\sigma}{\sqrt{2\pi}}$$

Where ϕ is the angle between the two half-lines.
So far, so good. Part of the general expression for all t , the piece between $\{\,\}$, is considered once again:

$$f(t) = \frac{\sigma}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2/\sigma^2} \;\pm\; t\;Erf(\pm t/\sigma)$$

When differentiating to t , it becomes:

$$f'(t) = \frac{\sigma}{\sqrt{2\pi}}\frac{-t}{\sigma^2}e^{-\frac{1}{2}t^2/\sigma^2} \;\pm\; t\;\frac{\pm 1}{\sigma}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2/\sigma^2} \;\pm\; Erf(\pm t/\sigma)$$

$$\Longrightarrow \quad f'(t) = \pm\; Erf(\pm t/\sigma)$$

Thus the first order fuzzyfied derivative of the kinky line becomes:

$$\vec{r}\,'(t) = -Erf(-t/\sigma)\cdot\left[\begin{array}{c} cos(\alpha) \\ sin(\alpha) \end{array}\right] + Erf(t/\sigma)\cdot\left[\begin{array}{c} cos(\beta) \\ sin(\beta) \end{array}\right]$$

This converges to the directions of the original half-lines for $t \to \pm\infty$. Moreover we find, for the origin of the kink:

$$\vec{r}\,'(0) = \frac{1}{2}\left\{\left[\begin{array}{c} cos(\beta) \\ sin(\beta) \end{array}\right] - \left[\begin{array}{c} cos(\alpha) \\ sin(\alpha) \end{array}\right]\right\}$$

The second order derivative is a piece of cake now:

$$\vec{r}\,''(t) = \frac{1}{\sigma\sqrt{2\pi}}\;e^{-\frac{1}{2}t^2/\sigma^2}\left\{\left[\begin{array}{c} cos(\alpha) \\ sin(\alpha) \end{array}\right] + \left[\begin{array}{c} cos(\beta) \\ sin(\beta) \end{array}\right]\right\}$$

It is seen that this "acceleration" has the same direction everywhere, which is equal to the mean of the directions of the two half-lines.

Normal Behaviour

Consider again the (sightly modified) defining expression for a fuzzyfied contour. Introduce a new integrating variable $\tau - t = u$ and employ a Taylor series expansion around (t):

$$\begin{bmatrix} \overline{x}(t) \\ \overline{y}(t) \end{bmatrix} = \int_{-\infty}^{+\infty} \begin{bmatrix} x(\tau) \\ y(\tau) \end{bmatrix} \frac{1}{\sigma\sqrt{2\pi}} \; e^{-\frac{1}{2}(\tau-t)^2/\sigma^2} d\tau =$$

$$\int_{-\infty}^{+\infty} \begin{bmatrix} x(t+u) \\ y(t+u) \end{bmatrix} \frac{1}{\sigma\sqrt{2\pi}} \; e^{-\frac{1}{2}u^2/\sigma^2} du =$$

$$\int_{-\infty}^{+\infty} \sum_{k=0}^{\infty} \left\{ \begin{bmatrix} x^{(k)}(t) \\ y^{(k)}(t) \end{bmatrix} \frac{u^k}{k!} \right\} \frac{1}{\sigma\sqrt{2\pi}} \; e^{-\frac{1}{2}u^2/\sigma^2} du$$

Exchange integration and summation:

$$= \sum_{k=0}^{\infty} \frac{1}{k!} \begin{bmatrix} x^{(k)}(t) \\ y^{(k)}(t) \end{bmatrix} \int_{-\infty}^{+\infty} u^k \frac{1}{\sigma\sqrt{2\pi}} \; e^{-\frac{1}{2}u^2/\sigma^2} du$$

Subsequent momenta are recognized:

$$\int_{-\infty}^{+\infty} u^0 \frac{1}{\sigma\sqrt{2\pi}} \; e^{-\frac{1}{2}u^2/\sigma^2} du = 1 \quad \text{and} \quad \int_{-\infty}^{+\infty} u^1 \frac{1}{\sigma\sqrt{2\pi}} \; e^{-\frac{1}{2}u^2/\sigma^2} du = 0$$

$$\int_{-\infty}^{+\infty} u^2 \frac{1}{\sigma\sqrt{2\pi}} \; e^{-\frac{1}{2}u^2/\sigma^2} du = \sigma^2 \quad \text{and} \quad \int_{-\infty}^{+\infty} u^3 \frac{1}{\sigma\sqrt{2\pi}} \; e^{-\frac{1}{2}u^2/\sigma^2} du = 0$$

At last, employ the arc-length $t = s$ as the running parameter. Then the series is approximated up to an order $O(\Delta s^4)$ by:

$$\begin{bmatrix} \overline{x}(s) \\ \overline{y}(s) \end{bmatrix} \approx \begin{bmatrix} x(s) \\ y(s) \end{bmatrix} + \frac{1}{2}\sigma^2 \begin{bmatrix} x''(s) \\ y''(s) \end{bmatrix} = \vec{r}(s) + \frac{1}{2}\rho\,\sigma^2\,\vec{n}$$

Where it is assumed that the contour exhibits a rather *smooth* behaviour at these points. (We have seen that *kinks* are a different matter altogether.)
The above result is confirmed by another finding, provided that it may assumed for a moment that the theory of *Lissajous / Fourier* expansions is already well known. It can be established very easily that the Fourier expansion of a circle with radius R is simply given by:

$$\begin{bmatrix} x(s) \\ y(s) \end{bmatrix} = \begin{bmatrix} x_m \\ y_m \end{bmatrix} + R \cdot \begin{bmatrix} cos(\phi) \\ sin(\phi) \end{bmatrix}$$

Here ϕ is the angle and $\phi = s.2\pi/L$ where s is the partial arc length and $L = 2\pi.R$ is the total arc length. The *fuzzyfication* of such a Lissajous expansion is given, in general, by:

$$\begin{bmatrix} \overline{x}(t) \\ \overline{y}(t) \end{bmatrix} = \sum_{k=0}^{\infty} e^{-\frac{1}{2}(\sigma k\omega)^2} \begin{bmatrix} a_x(k)\; cos(k\omega t) + b_x(k)\; sin(k\omega t) \\ a_y(k)\; cos(k\omega t) + b_y(k)\; sin(k\omega t) \end{bmatrix}$$

In our case $L = 2\pi.R \rightarrow \omega = 1/R$ and $k = 0, 1$. Thus making that the general expression is greatly simplified:

$$\begin{bmatrix} \overline{x}(t) \\ \overline{y}(t) \end{bmatrix} = \begin{bmatrix} x_m \\ y_m \end{bmatrix} + e^{-\frac{1}{2}(\sigma/R)^2} R \cdot \begin{bmatrix} cos(\phi) \\ sin(\phi) \end{bmatrix}$$

With other words, the fuzzyfied circle *shrinks* with an amount $exp(-(\sigma/R)^2/2)$ when compared with the original one. The first few terms of the accompanying Taylor Series expansion are:

$$\begin{bmatrix} \overline{x}(t) \\ \overline{y}(t) \end{bmatrix} \approx \begin{bmatrix} x_m \\ y_m \end{bmatrix} + R \cdot \begin{bmatrix} cos(\phi) \\ sin(\phi) \end{bmatrix} - R.\frac{1}{2}(\sigma/R)^2 \cdot \begin{bmatrix} cos(\phi) \\ sin(\phi) \end{bmatrix}$$

If (x_m, y_m, R) is the osculating circle of a curve, then it is augmented by a term:

$$-\frac{1}{2}\frac{1}{R}\sigma^2 = +\frac{1}{2}\rho\sigma^2$$

Which is quite in agreement with the results found before.